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Duality invariance of non–anticommutative $\mathrm{N} = \frac{1}{2}$ supersymmetric $\mathrm{U}(1)$ gauge theory

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Abstract: A parent action is introduced to formulate (S–) dual of non–anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory. Partition function for parent action in phase space is utilized to establish the equivalence of partition functions of the theories which this parent action produces. Thus, duality invariance of non–anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory follows. The results which we obtained are valid at tree level or equivalently at the first order in the nonanticommutativity parameter $C_{\mu\nu}$.

Keywords: S–duality, Supersymmetry, Noncommutativity.

Contents

1. Introduction

The formalism of superstring theory with pure spinors[1] in a graviphoton background[2] gives rise to a non-anticommutative superspace[3],[4] which was introduced also in other contexts [5], [6]. Moyal antibrackets (star products) are employed to interpose non-anticommutativity between the coordinates. Thus, instead of coordinates which are operators, one deals with the usual superspace variables. Vector superfields taking values in this deformed superspace utilized to define a non–anticommutative supersymmetric Yang-Mills gauge theory. However, due to a change of variables one deals with the standard gauge transformations and component fields[3]. Deformation of 4 dimensional $N = 1$ superspace by making the chiral fermionic coordinates θ_α, $\alpha = 1, 2$, non–anticommuting

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta},$$

where $C^{\alpha\beta}$ are constant deformation parameters, breaks half of the supersymmetry[3]. In euclidean $\mathbb{R}^4$ chiral and antichiral fermions are not related with complex conjugation. The vector superfield of this deformed superspace was employed to derive, after a change of variables, the $N = \frac{1}{2}$ supersymmetric Yang-Mills theory action[3][1]

$$I_{1/2} = \frac{1}{g^2} \int d^4x Tr \left\{ -\frac{1}{4} G^{\mu\nu} G_{\mu\nu} - i\lambda \not{\mathcal{D}} \bar{\lambda} + \frac{1}{2} D^2 - \frac{i}{2} C^{\mu\nu} G_{\mu\nu} (\bar{\lambda}\bar{\lambda}) + \frac{|C|^2}{8} (\bar{\lambda}\bar{\lambda})^2 \right\}, \quad (1.1)$$

where $C^{\mu\nu} = C^{\alpha\beta} \epsilon_{\beta\gamma} \sigma^{\mu\nu\ \gamma}_{\alpha}$ and $\mathcal{D}_\mu$ is the covariant derivative. Gauge transformations possess the usual form. $G_{\mu\nu}$ is the non–abelian field strength related to the gauge field A_μ. λ , $\bar{\lambda}$ are independent fermionic fields and D is auxiliary bosonic field. Although we deal

[1] For another approach see [7].

with euclidean $\mathbb{R}^4$, we use Minkowski space notation and follow the conventions of [8]. The surviving part of the $N = 1$ supersymmetry acts on the standard component fields as

$$\begin{aligned} \delta\lambda &= i\epsilon D + \sigma^{\mu\nu}\epsilon(G_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\bar{\lambda}\bar{\lambda}) \\ \delta A_\mu &= -i\bar{\lambda}\bar{\sigma}_\mu\epsilon \\ \delta D &= -\epsilon\sigma^\mu\mathcal{D}_\mu\bar{\lambda} \\ \delta\bar{\lambda} &= 0\ , \end{aligned} \tag{1.2}$$

where ϵ is a constant Grassmann parameter. The action (1.1) can also be obtained by applying the supersymmetry generator Q defined by $\delta = \epsilon Q$, to the lower dimensional field monomial $Tr\lambda\lambda$ as

$$I_{1/2} = \frac{1}{8g^2}Q^2\int d^4x Tr(\lambda\lambda), \tag{1.3}$$

up to total derivatives, similar to the usual $N = 1$ super Yang-Mills theory[9].

(S–) Duality transformations map strong coupling domains to weak coupling domains of gauge theories. Although duality invariance of pure $U(1)$ gauge theory can be shown, trivially, by rescaling its gauge fields[2], it can also be studied in terms of parent action formalism[10]. The latter approach permits to introduce a dual formulation of the non-commutative $U(1)$ gauge theory[11]. Moreover, dual actions for supersymmetric $U(1)$ gauge theory have already been derived utilizing a parent action when only bosonic coordinates are noncommuting[12]. Actually, (S–) duality is helpful for inverting computations performed in weak coupling domains to strong coupling domains, when partition functions of the "original" and dual theories are equivalent, i.e. when there exists duality symmetry. For noncommutative $U(1)$ gauge theory without supersymmetry, this equivalence was established within the hamiltonian formalism[13].

We would like to investigate duality properties of the $N = \frac{1}{2}$ supersymmetric non–anticommutative theory (1.1) with $U(1)$ gauge group. We will define dual theory by introducing a parent action which produces the original theory when "dual" fields are eliminated by their equations of motion. Hamiltonian formalism of parent action is used to construct its partition function in phase space. We show that this partition function gives rise to either partition function of the original $N = \frac{1}{2}$ superysmmetric non–anticommutative $U(1)$ gauge theory or partition function of its dual theory. Then, we conclude that the $N = \frac{1}{2}$ supersymmetric non–anticommutative $U(1)$ gauge theory is duality invariant. We do not consider loops, so that our results are valid at tree level which is equivalent to the first order approximation in $C_{\mu\nu}$.

In Section 2 we introduce a parent action. We show that it generates non-anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory. Then, we obtain obtain the dual theory resulting from this parent action. In Section 3 we present hamiltonian formulation of the original and dual non-anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory. In Section 4, we first exhibit constrained hamiltonian structure arising from the parent action.

[2] For $U(1)$ gauge theory rescaling $A \to g^2 A_D$ results in the duality transformation $g^{-2}\int dA\wedge dA \to g^2\int dA_D \wedge dA_D$.

Then, its path integral in the phase space is presented. By integrating over the appropriate variables we demonstrate the equivalence of partition functions of dual and original non-anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theories. Lastly we comment on quantum corrections.

2. Dual of non–anticommutative $\mathbf{N = \frac{1}{2}}$ supersymmetric $\mathbf{U(1)}$ gauge theory

Parent action of supersymmetric $U(1)$ gauge theory by superfields was given in [14]. Once written in terms of component fields it was generalized to provide dual formulations of noncommutative supersymmetric $U(1)$ gauge theory when only bosonic coordinates of superspace are mutually noncommuting[12]. By a similar approach we would like to introduce a parent action for the non–anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory obtained from (1.1). We propose the following parent action in terms of component fields $\mathrm{X} = (F_{\mu\nu},\ \lambda_\alpha,\ \bar{\lambda}^{\dot{\alpha}},\ \psi_\alpha,\ \bar{\psi}^{\dot{\alpha}},\ D_1,\ D_2)$ and $\mathrm{X}_D = (A_{D\mu},\ \lambda_{D\alpha},\ \bar{\lambda}^{\dot{\alpha}}_D,\ D_D)$,

$$I_p = I_0[\mathrm{X}] + I_l[\mathrm{X}, \mathrm{X}_D] \tag{2.1}$$

where I_0 is suggested by (1.1)

$$I_0 = \frac{1}{g^2}\int d^4x \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{2}\lambda\partial\!\!\!/\bar{\lambda} - \frac{i}{2}\bar{\psi}\bar{\partial\!\!\!/}\psi + \frac{1}{4}D_1^2 + \frac{1}{4}D_2^2 - \frac{i}{4}C^{\mu\nu}F_{\mu\nu}(\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \right\} \tag{2.2}$$

and I_l is defined as

$$I_l = \int d^4x \left\{ \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\mu\nu}\partial_\lambda A_{D\kappa} + \frac{1}{2}\lambda\partial\!\!\!/\bar{\lambda}_D + \frac{1}{2}\lambda_D\partial\!\!\!/\bar{\lambda} - \frac{1}{2}\bar{\psi}\bar{\partial\!\!\!/}\lambda_D - \frac{1}{2}\bar{\lambda}_D\bar{\partial\!\!\!/}\psi + \frac{i}{2}D_D(D_1 - D_2) \right\}. \tag{2.3}$$

Here $F_{\mu\nu}$ are independent field variables which are not associated with any gauge field.

The equations of motion with respect to the "dual" fields X_D are

$$\epsilon^{\mu\nu\lambda\kappa}\partial_\nu F_{\lambda\kappa} = 0\ , \tag{2.4}$$

$$\partial\!\!\!/\bar{\psi} = \partial\!\!\!/\bar{\lambda} \quad , \quad \bar{\partial\!\!\!/}\psi = \bar{\partial\!\!\!/}\lambda, \quad , \quad D_1 = D_2 = D\ . \tag{2.5}$$

One solves (2.4) by setting $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which is the field strength of the gauge field A_μ. When one plugs this and the solutions of the other equations of motion (2.5) in terms of $\lambda,\ \bar{\lambda},\ D$, into the parent action (2.1), the non–anticommuting $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory action follows:

$$I = \frac{1}{g^2}\int d^4x \left\{ -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - i\lambda\partial\!\!\!/\bar{\lambda} + \frac{1}{2}D^2 - \frac{i}{2}C^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)\bar{\lambda}\bar{\lambda} \right\}. \tag{2.6}$$

Since we deal with $U(1)$ gauge group, the term quadratic in the deformation parameter, $\frac{|C|^2}{8}(\bar{\lambda}\bar{\lambda})^2$, of the action (1.1) vanishes.

On the other hand, the equations of motion with respect to the fields X are

$$
\begin{aligned}
&\frac{1}{2g^2}F^{\mu\nu}+\frac{i}{4g^2}C^{\mu\nu}(\bar\lambda\bar\lambda+\bar\psi\bar\psi)-\frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}\partial_\lambda A_{D_\kappa}=0\ ,\\
&\partial\!\!\!/\bar\lambda+ig^2\partial\!\!\!/\bar\lambda_D=0\quad,\quad \partial\!\!\!/\bar\psi-ig^2\partial\!\!\!/\bar\lambda_D=0\ ,\\
&\bar\partial\!\!\!/\lambda+C^{\mu\nu}F_{\mu\nu}\bar\lambda+ig^2\bar\partial\!\!\!/\lambda_D=0\quad,\quad \bar\partial\!\!\!/\psi+C^{\mu\nu}F_{\mu\nu}\bar\psi-ig^2\bar\partial\!\!\!/\lambda_D=0\ ,\\
&D_1+ig^2D_D=0\quad,\quad D_2-ig^2D_D=0
\end{aligned}
\tag{2.7}
$$

where $F_{D\mu\nu}=\partial_\mu A_{D\nu}-\partial_\nu A_{D\mu}$. We solve the equations of motion (2.7) for X fields in terms of X_D and substitute them in the parent action (2.1) to obtain the dual non–anticommutative $N=\frac{1}{2}$ supersymmetric $U(1)$ gauge theory action :

$$
I_D=g^2\int d^4x\Big\{-\frac{1}{4}F_D^{\mu\nu}F_{D\mu\nu}-i\lambda_D\partial\!\!\!/\bar\lambda_D+\frac{1}{2}D_D^2+\frac{i}{4}g^2\epsilon^{\mu\nu\lambda\kappa}C_{\mu\nu}F_{D\lambda\kappa}\bar\lambda_D\bar\lambda_D\Big\}. \tag{2.8}
$$

One can observe that the non–anticommutative $N=\frac{1}{2}$ supersymmetric $U(1)$ gauge theory action (2.6) and its dual (2.8) possess the same form and

$$
\begin{aligned}
&g\rightarrow\frac{1}{g}\\
&C^{\mu\nu}\rightarrow C_D^{\mu\nu}=-\frac{1}{2}g^2\epsilon^{\mu\nu\lambda\kappa}C_{\lambda\kappa}=ig^2C^{\mu\nu}
\end{aligned}
\tag{2.9}
$$

is the duality transformation.

3. Hamiltonian formulations of non–anticommutative $\mathbf{N=\frac{1}{2}}$ supersymmetric $\mathbf{U(1)}$ gauge theory and its dual

To acquire hamiltonian formulation of the non–anticommutative $N=\frac{1}{2}$ supersymmetric $U(1)$ gauge theory (2.6), let us introduce the canonical momenta $(P^\mu,\ \Pi^\alpha,\ \bar\Pi_{\dot\alpha},\ P)$ corresponding to $(A_\mu,\ \lambda_\alpha,\ \bar\lambda^{\dot\alpha},\ D)$. Canonical momenta associated with A_i, $i=1,2,3$; are

$$
P^i=-\frac{1}{g^2}(\partial^0A^i-\partial^iA^0)-\frac{i}{g^2}C^{0i}\lambda\lambda. \tag{3.1}
$$

However, definitions of the other momenta lead to the weakly vanishing primary constraints[3],

$$
\begin{aligned}
&\phi_1\equiv P^0\approx 0\quad,\quad \Phi_1\equiv P\approx 0\\
&\chi^\alpha\equiv\Pi^\alpha\approx 0\quad,\ \bar\chi_{\dot\alpha}\equiv\bar\Pi_{\dot\alpha}-\frac{i}{g^2}\lambda^\alpha\sigma^0_{\alpha\dot\alpha}\approx 0\ .
\end{aligned}
\tag{3.2}
$$

Canonical hamiltonian associated with the action (2.6) is derived to be

$$
\begin{aligned}
\mathcal{H}_c=&\frac{g^2}{2}P_i^2+\frac{1}{g^2}\{\frac{1}{4}(\partial_iA_j-\partial_jA_i)^2+i\lambda\nabla\!\!\!\!/\,\bar\lambda-\frac{1}{2}D^2+\frac{i}{2}C^{ij}(\partial_iA_j-\partial_jA_i)\bar\lambda\bar\lambda\}\\
&-A_0\partial_iP^i-iC_{0i}P^i\bar\lambda\bar\lambda
\end{aligned}
\tag{3.3}
$$

[3]We use left derivatives with respect to anticommuting variables throughout this work.

where $\bar{\nabla} = \sigma^i \partial_i$.

Let the primary constraints (3.2) be collectively denoted as Θ^a. Then the extended hamiltonian is given by

$$\mathcal{H}_E = \mathcal{H}_c + l_a \Theta^a \ , \tag{3.4}$$

where l_a are Lagrange multipliers. Consistency of the primary constraints (3.2) with the equations of motion following from (3.4):

$$\dot{\Theta}^a = \{\mathcal{H}_E, \Theta^a\} \approx 0$$

gives rise to the secondary constraints

$$\phi_2 \equiv \partial_i P_i \approx 0 \quad , \quad \Phi_2 \equiv D \approx 0 \ . \tag{3.5}$$

There are no other constraints arising from these secondary constraints. One can show that $\phi_1,\ \phi_2$ are first class and $\Phi_1,\ \Phi_2,\ \chi^\alpha,\ \bar{\chi}_{\dot\alpha}$ are second class constraints.

Hamiltonian structure of the dual theory (2.8) is similar to (3.2)–(3.5). Indeed, canonical hamiltonian associated with the dual action (2.8) can easily be read from (3.3) as

$$\begin{aligned} \mathcal{H}_{Dc} = {} & \frac{1}{2g^2} P_D^i P_{Di} + g^2 \{ \frac{1}{4} F_D^{ij} F_{Dij} + i \lambda_D \bar{\nabla} \bar{\lambda}_D - \frac{1}{2} D_D^2 + \frac{i}{2} C_D^{ij} F_{ij} \bar{\lambda}_D \bar{\lambda}_D \} \\ & - A_{D0} \partial_i P_D^i - i C_{D0i} P_D^i \bar{\lambda}_D \bar{\lambda}_D . \end{aligned} \tag{3.6}$$

Moreover, there are the hamiltonian constraints which can be obtained from (3.2) and (3.5) by replacing $(P^0,\ \Pi^\alpha,\ \bar{\Pi}_{\dot\alpha},\ P,\ A_i,\ \lambda_\alpha,\ D)$ with $(P_D^0,\ \Pi_D^\alpha,\ \bar{\Pi}_{D\dot\alpha},\ P_D,\ A_{Di},\ g^4 \lambda_{D\alpha},\ D_D)$.

4. Equivalence of partition functions for non–anticommutative $\mathbf{N = \frac{1}{2}}$ supersymmetric $\mathbf{U(1)}$ theory and its dual

Partition function for the parent action (2.1) is expected to produce partition functions of the actions (2.6) and (2.8). In (2.1) there are some terms cubic in fields. Thus, it would be apposite to discuss its partition function in phase space, where integrations would be simplified due to hamiltonian constraints. To achieve hamiltonian formulation let us introduce the set of canonical momenta $(P^{\mu\nu},\ \Pi_1^\alpha,\ \bar{\Pi}_{1\dot\alpha},\ \Pi_2^\alpha,\ \bar{\Pi}_{2\dot\alpha},\ P_1,\ P_2)$ and $(P_D^\mu,\ \Pi_D^\alpha,\ \bar{\Pi}_{D\dot\alpha},\ P_D)$ corresponding to the fields $(F_{\mu\nu},\ \lambda_\alpha,\ \bar{\lambda}^{\dot\alpha},\ \psi_\alpha,\ \bar{\psi}^{\dot\alpha},\ D_1,\ D_2)$ and to the dual variables $(A_{D\mu},\ \lambda_{D\alpha},\ \bar{\lambda}_D^{\dot\alpha},\ D_D)$. Each of the canonical momenta resulting from the parent action (2.1) gives rise to a primary constraint, which we collectively denote them as $\{\Theta^a\}$:

$$\begin{aligned} \phi_1^{0i} \equiv P^{0i} \approx 0, \qquad & \phi_2^{ij} \equiv P^{ij} \approx 0 \ , \\ \chi_1^\alpha \equiv \Pi_1^\alpha \approx 0, \qquad & \bar{\chi}_{1\dot\alpha} \equiv \bar{\Pi}_{1\dot\alpha} - \frac{i}{2g^2} \lambda^\alpha \sigma^0_{\alpha\dot\alpha} + \frac{1}{2} \lambda_D^\alpha \sigma^0_{\alpha\dot\alpha} \approx 0, \\ \chi_2^\alpha \equiv \Pi_2^\alpha - \frac{i}{2g^2} \bar{\psi}_{\dot\alpha} \bar{\sigma}^{0\dot\alpha\alpha} - \frac{1}{2} \bar{\lambda}_{D\dot\alpha} \bar{\sigma}^{0\dot\alpha\alpha} \approx 0, \qquad & \chi_{2\dot\alpha} \equiv \bar{\Pi}_{2\dot\alpha} \approx 0, \\ \Phi_1 \equiv P_1 \approx 0, \qquad & \Phi_2 \equiv P_2 \approx 0, \\ \phi_{D_1} \equiv P_D^0 \approx 0, \qquad & \phi_{D_2}^i \equiv P_D^i - \frac{1}{2} \epsilon^{ijk} F_{jk} \approx 0, \end{aligned}$$

$$\chi_D^\alpha \equiv \Pi_D^\alpha - \frac{1}{2}\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{0\dot{\alpha}\alpha} \approx 0, \qquad \chi_{D\dot{\alpha}} \equiv \bar{\Pi}_{D\dot{\alpha}} + \frac{1}{2}\lambda^\alpha \sigma^0_{\alpha\dot{\alpha}} \approx 0,$$
$$\Phi_D \equiv P_D \approx 0. \tag{4.1}$$

Canonical hamiltonian associated with the parent action (2.1) is then found to be

$$\begin{aligned}\mathcal{H}_p = & \frac{1}{g^2}\Big[\frac{1}{4}F_{\mu\nu}^2 + \frac{i}{2}\lambda\nabla\!\!\!\!/\,\bar{\lambda} + \frac{i}{2}\bar{\psi}\bar{\nabla\!\!\!\!/}\psi - \frac{1}{4}(D_1^2 + D_2^2) + \frac{i}{4}C^{\mu\nu}F_{\mu\nu}(\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi})\Big] \\ & -\epsilon^{ijk}F_{0i}\partial_j A_{D\kappa} + \frac{1}{2}\epsilon^{ijk}F_{ij}\partial_\kappa A_{D0} - \frac{1}{2}\lambda\nabla\!\!\!\!/\,\bar{\lambda}_D - \frac{1}{2}\lambda_D\nabla\!\!\!\!/\,\bar{\lambda} + \frac{1}{2}\bar{\psi}\bar{\nabla\!\!\!\!/}\lambda_D \\ & + \frac{1}{2}\bar{\lambda}_D\bar{\nabla\!\!\!\!/}\psi - \frac{i}{2}D_D(D_1 - D_2).\end{aligned} \tag{4.2}$$

Extended hamiltonian is obtained by adding the primary constraints Θ^a with the help of Lagrange multipliers l_a, to the canonical hamiltonian (4.2):

$$\mathcal{H}_E = \mathcal{H}_p + l_a\Theta^a \tag{4.3}$$

Consistency of the primary constraints with the equations of motion:

$$\dot{\Theta}^a = \{\mathcal{H}_E, \Theta^a\} \approx 0$$

gives rise to the secondary constraints

$$\begin{aligned}\Delta_1 &\equiv \{\mathcal{H}_p, P_1\} = -\frac{1}{2g^2}D_1 - \frac{i}{2}D_D \approx 0\ , \\ \Delta_2 &\equiv \{\mathcal{H}_p, P_2\} = -\frac{1}{2g^2}D_2 + \frac{i}{2}D_D \approx 0\ , \\ \Delta_D &\equiv \{\mathcal{H}_p, P_D\} = \frac{i}{2}(D_1 - D_2) \approx 0\ , \\ \varphi_D &\equiv \{\mathcal{H}_p, P_D^0\} = \frac{1}{2}\epsilon^{ijk}\partial_k F_{ij} \approx 0\ , \\ \varphi_1^{0i} &\equiv \{\mathcal{H}_p, P_{0i}\} = F^{0i} - g^2\epsilon^{ijk}\partial_j A_{Dk} + \frac{ig^2}{2}C^{0i}(\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \approx 0\ .\end{aligned} \tag{4.4}$$

In path integrals first and second class constraints are treated on different grounds. Thus, let us first identify the first class constraints: ϕ_{D1} is obviously first class. Moreover, we observe that the linear combination

$$\phi_{D_3} \equiv \partial_i\phi^i_{D_2} + \varphi_D = \partial_i P^i_D \approx 0, \tag{4.5}$$

is also a first class constraint. There are no other first class constraints. However, the constraints $\phi^i_{D_2}$ contain second class constraints which we should separate out. This is due to the fact that a vector can be completely described by giving its divergence and rotation (up to a boundary condition). (4.5) is derived taking divergence of $\phi^i_{D_2}$, so that, there are still two linearly independent second class constraints following from the curl of $\phi^i_{D_2}$:

$$\phi^n_{D_4} \equiv K^n_i\phi^i_{D_2} = \mathcal{K}^{ni}\epsilon_{ijk}\partial^j\phi^k_{D_2} \approx 0, \tag{4.6}$$

where $n = 1, 2$. $\mathcal{K}_i^n$ are some constants whose explicit forms are not needed for the purposes of this work. Although all of them are second class, we would like to separate φ_1^{0i} in a similar manner:

$$\varphi_2 \equiv \partial_i \varphi_1^{0i} = -\partial_i F^{0i} - \frac{i}{2} C^{0i} \partial_i (\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \approx 0, \tag{4.7}$$

$$\varphi_3^n \equiv L_i^n \varphi_1^{0i} = \mathcal{L}^{ni} \epsilon_{ijk} \partial^j \varphi_1^{0k} \approx 0. \tag{4.8}$$

where $\mathcal{L}^{nj}$ are some constants. The reason of preferring this set of constraints will be clear when we perform the path integrals, though explicit forms of $\mathcal{L}_i^n$ play no role in our calculations.

In phase space, partition function can be written as[15],[16]

$$\mathcal{Z} = \int \prod_i \mathcal{D}\Upsilon_i \, \mathcal{D}P_{\Upsilon_i} \, \mathcal{M} \, e^{i \int d^3x (\dot{\Upsilon}_i P_{\Upsilon_i} - \mathcal{H}_p)} \tag{4.9}$$

$$\mathcal{M} = N \det(\partial_i^2) \delta(\partial \cdot \mathbf{P}_D) \delta(\partial \cdot \mathbf{A}_D) \delta(P_{D0}) \delta(A_{D0}) \mathrm{sdet}\, M \prod_z \delta(S_z), \tag{4.10}$$

where Υ_i and P_{Υ_i} embrace all of the fields and their momenta. S_z denotes all second class constraints: $S_z \equiv (\phi_1, \phi_2, \Phi_1, \Phi_2, \phi_{D_4}, \Phi_D, \varphi_2, \varphi_3, \Delta_1, \Delta_2, \varphi_D, \Delta_D, \chi_1, \bar{\chi}_1, \chi_2, \bar{\chi}_2, \chi_D, \bar{\chi}_D)$. We adopted the gauge fixing (auxiliary) conditions

$$\begin{aligned} A_{D_0} &= 0 \,, \\ \partial_i A_{D_i} &= 0, \end{aligned} \tag{4.11}$$

for the first class constraints ϕ_{D_1} and ϕ_{D_3}. N is a normalization constant. The matrix of the generalized Poisson brackets of the second class constraints $M = \{S_z, S_{z'}\}$ can be written in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{4.12}$$

so that, its superdeterminant is given by

$$\mathrm{sdet} M = (\det D)^{-1} \det(A - BD^{-1}C). \tag{4.13}$$

Calculations of B, C and D can be shown to yield

$$(BD^{-1}C) = 0.$$

Therefore, (4.13) is simplified as

$$\mathrm{sdet} M = \frac{\det A}{\det D}. \tag{4.14}$$

Contribution of fermionic constraints is

$$\det D^{-1} = -(4 \det g^2)^2. \tag{4.15}$$

Here, $\det g^2$, which arise because we deal with constraints of a field theory, should appropriately be regularized. Contribution of the bosonic constraints has already been calculated in [13]:

$$\det A = \det\left(\epsilon_{ijk}\partial^i K_1^j K_2^k\right)\det\left(\epsilon_{ijk}\partial^i L_1^j L_2^k\right). \tag{4.16}$$

The linear operators K_i^n and L_i^n are defined in (4.6) and (4.8). These determinants which are multiplication of three linear operators should be interpreted as multiplication of their eigenvalues.

In (4.9) the integrals over all of the fermionic momenta and $P_{\mu\nu}$ can be easily performed utilizing the related delta functions, to get

$$\begin{aligned}
\mathcal{Z} = &\int \mathcal{D}F^{\mu\nu}\,\mathcal{D}\lambda\,\mathcal{D}\psi\,\mathcal{D}\bar{\lambda}\,\mathcal{D}\bar{\psi}\,\mathcal{D}D_1\,\mathcal{D}P_1\,\mathcal{D}D_2\,\mathcal{D}P_2\,\mathcal{D}A_{D\mu}\,\mathcal{D}\lambda_D\,\mathcal{D}\bar{\lambda}_D\,\mathcal{D}P_{D\mu}\,\mathcal{D}D_D\,\mathcal{D}P_D \\
&\tilde{\mathcal{M}}\exp\Bigg\{i\int d^3x\Bigg[P_1\dot{D}_1 + P_2\dot{D}_2 + P_D^0\dot{A}_{D0} + P_D^i\dot{A}_{Di} + P_D\dot{D}_D - \frac{1}{4g^2}F^{0i}F_{0i} \\
&-\frac{1}{4g^2}F^{ij}F_{ij} - \frac{i}{2g^2}\lambda\partial\!\!\!/\bar{\lambda} - \frac{i}{2g^2}\bar{\psi}\bar{\partial\!\!\!/}\psi + \frac{1}{4g^2}(D_1^2 + D_2^2) - \frac{i}{2g^2}C^{0i}F_{0i}(\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) \\
&-\frac{i}{4g^2}C^{ij}F_{ij}(\bar{\lambda}\bar{\lambda} + \bar{\psi}\bar{\psi}) + \epsilon^{ijk}F_{0i}\partial_j A_{Dk} - \frac{1}{2}\epsilon^{ijk}F_{ij}\partial_k A_{D0} \\
&+\frac{1}{2}\lambda\partial\!\!\!/\bar{\lambda}_D + \frac{1}{2}\lambda_D\partial\!\!\!/\bar{\lambda} - \frac{1}{2}\bar{\psi}\bar{\partial\!\!\!/}\lambda_D - \frac{1}{2}\bar{\lambda}_D\bar{\partial\!\!\!/}\psi + \frac{i}{2}D_D(D_1 - D_2)\Bigg]\Bigg\}.
\end{aligned} \tag{4.17}$$

Here, $\tilde{\mathcal{M}}$ is the same with $\mathcal{M}$ except the delta functions which we utilized above. We first would like to integrate over the fields which do not carry the label "D" : P_1, P_2 integrals are trivially performed and by integrating over D_1 and D_2 we get a factor of $\det g^2$ and $\delta(D_D)$. Integrations over ψ and λ yield $(\det\partial\!\!\!//\det g^2)^2\delta(i\bar{\psi} + g^2\bar{\lambda}_D)\delta(i\bar{\lambda} - g^2\bar{\lambda}_D)$. Thus, we replace $\bar{\psi}$ with $ig^2\bar{\lambda}_D$ and $\bar{\lambda}$ with $-ig^2\bar{\lambda}_D$ after integrating over $\bar{\psi}$ and $\bar{\lambda}$. Integrations over $F^{\mu\nu}$ yield substitution of F^{0i} with $g^2\epsilon^{ijk}\partial_j A_{Dk} + \frac{i}{2}g^4C^{0i}\bar{\lambda}_D\bar{\lambda}_D$, F^{ij} with $\epsilon^{ijk}P_{Dk}$ and cancellation of the determinant (4.16). Moreover, we integrate over A_D^0, P_D^0 and choose the normalization constant N such that we get

$$\begin{aligned}
\mathcal{Z} = &\int \mathcal{D}A_{Di}\,\mathcal{D}\bar{\lambda}_D\,\mathcal{D}P_{Di}\,\mathcal{D}D_D\,\mathcal{D}P_D\,(\det g^2)\det\partial_i^2)(\det\partial\!\!\!/)^2\,\delta(D_D)\delta(P_D) \\
&\delta(\partial\cdot\mathbf{P}_D)\delta(\partial\cdot\mathbf{A}_D)\;\exp\Bigg\{i\int d^3x\Bigg[P_D^i\dot{A}_{Di} + P_D\dot{D}_D - \frac{1}{2g^2}P_{Di}P_D^i - iC_D^{0i}P_{Di}\bar{\lambda}_D\bar{\lambda}_D \\
&-\frac{g^2}{4}F_D^{ij}F_{Dij} - \frac{i}{2}g^2C_D^{ij}F_{Dij}\bar{\lambda}_D\bar{\lambda}_D - ig^2\lambda_D\partial\!\!\!/\bar{\lambda}_D + \frac{g^2}{2}D_D^2\Bigg]\Bigg\}.
\end{aligned} \tag{4.18}$$

In the exponent we distinguish the first order lagrangian of the dual theory (3.6) where Π_D^α and $\bar{\Pi}_{D\dot{\alpha}}$ are eliminated from the path integral by performing their integrations.

Now, in (4.17) let us integrate over the fields carrying the label "D": P_D integral is trivial. Integration over D_D contributes as $(\det g^2)\delta(D_1 + D_2)\delta(D_1 - D_2)$. Integrations of the fermionic variables λ_D and $\bar{\lambda}_D$ lead to $\delta(-\partial\!\!\!/\bar{\psi} + \bar{\partial\!\!\!/}\psi)\delta(\bar{\partial\!\!\!/}\lambda - \bar{\partial\!\!\!/}\psi)$. Due to the constraint $\varphi_D = 0$ we set

$$F_{ij} = \partial_i A_j - \partial_j A_i. \tag{4.19}$$

However, this replacement does not diminish the relevant number of physical phase space variables as it should be the case if the second class constraint φ_D has been taken properly into account. Therefore, we adopt the change of variables (4.19) with the replacement [13]

$$\mathcal{D}F_{ij}\delta(\epsilon^{klm}\partial_k F_{lm})\delta(K^i_n(P_{Di}+\frac{1}{2}\epsilon_{ijk}F^{jk}))\rightarrow \det(\partial^2)\mathcal{D}A_i\delta(\partial_j A^j)\delta\left(K^i_n(P_{Di}+\epsilon_{ijk}\partial^j A^k)\right). \tag{4.20}$$

Expressing A_{Di} and P_{Di} in terms of the fields $(A_i,\ F_{0i})$ by making use of the delta functions $\delta(K^n_i\phi^i_D)\delta(L^n_i\phi^{0i}_1)\delta(\partial\cdot\mathbf{P}_D)\delta(\partial\cdot\mathbf{A}_D)$ contributes to the measure with

$$\left[(\det g^2)^2\det(\partial^2)\det\left(\epsilon_{ijk}\partial^i K^j_1 K^k_2\right)\det\left(\epsilon_{ijk}\partial^i L^j_1 L^k_2\right)\right]^{-1}.$$

Hence, integrations over A_{Di} and P_{Di} in (4.17) can be performed to obtain

$$\begin{aligned}\mathcal{Z}=&\int\mathcal{D}A_i\,\mathcal{D}F_{0i}\mathcal{D}\lambda\,\mathcal{D}\bar\lambda\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\mathcal{D}D_1\,\mathcal{D}P_1\,\mathcal{D}D_2\,\mathcal{D}P_2\,(\det g^2)\det(\partial_i^2)\,\delta(\partial\cdot\mathbf{A})\\ &\delta(D_1+D_2)\delta(D_1-D_2)\,\delta(-\partial\!\!\!/\bar\psi+\partial\!\!\!/\bar\lambda)\,\delta(\bar\partial\!\!\!/\lambda-\bar\partial\!\!\!/\psi)\,\delta\left(\partial_i F^{0i}+\frac{i}{2}\partial_i C^{0i}(\bar\lambda\bar\lambda+\bar\psi\bar\psi)\right)\\ &\exp\left\{i\int d^3x\left[\frac{1}{g^2}\left(F^{0i}+\frac{i}{4}C^{0i}(\bar\lambda\bar\lambda+\bar\psi\bar\psi)\right)\dot A_i+\dot D_1 P_1+\dot D_2 P_2-\frac{1}{2g^2}F^{0i}F_{0i}\right.\right.\\ &-\frac{1}{4g^2}(\partial_i A_j-\partial_j A_i)^2-\frac{i}{2g^2}\lambda\partial\!\!\!/\bar\lambda-\frac{i}{2g^2}\bar\psi\bar\partial\!\!\!/\psi+\frac{1}{4g^2}(D_1^2+D_2^2)\\ &\left.\left.-\frac{i}{2g^2}C^{0i}F_{0i}(\bar\lambda\bar\lambda+\bar\psi\bar\psi)-\frac{i}{4g^2}C^{ij}(\partial_i A_j-\partial_j A_i)(\bar\lambda\bar\lambda+\bar\psi\bar\psi)\right]\right\}\end{aligned} \tag{4.21}$$

Integrating over D_2, P_2, ψ, $\bar\psi$ and renaming $D_1=D$ and $P_1=P$ yield

$$\begin{aligned}\mathcal{Z}=&\int\mathcal{D}A_i\,\mathcal{D}F_{0i}\mathcal{D}\lambda\,\mathcal{D}\bar\lambda\,\mathcal{D}D\,\mathcal{D}P\,(\det g^2)(\det\partial_i^2)(\det\partial\!\!\!/)^2\delta(P)\delta(D)\delta(\partial\cdot\mathbf{A})\\ &\delta\left(\partial_i F^{0i}+i\partial_i C^{0i}\bar\lambda\bar\lambda\right)\exp\left\{i\int d^3x\left[\frac{1}{g^2}\left(F^{0i}+iC^{0i}\bar\lambda\bar\lambda\right)\dot A_i+\dot D P-\frac{1}{2g^2}F^{0i}F_{0i}\right.\right.\\ &\left.\left.-\frac{1}{4g^2}(\partial_i A_j-\partial_j A_i)^2-\frac{1}{g^2}\lambda\partial\!\!\!/\bar\lambda+\frac{1}{2g^2}D^2-\frac{i}{g^2}C^{ij}(\partial_i A_j-\partial_j A_i)\bar\lambda\bar\lambda\right]\right\}.\end{aligned} \tag{4.22}$$

In terms of the change of variables

$$\begin{aligned}g^2P^i&=F^{0i}+C^{0i}\bar\lambda\bar\lambda,\\ \mathcal{D}F^{0i}&=(\det g^2)\mathcal{D}P^i,\end{aligned} \tag{4.23}$$

we write the partition function (4.22) as

$$\begin{aligned}\mathcal{Z}=&\int\mathcal{D}A_i\,\mathcal{D}P^i\,\mathcal{D}\lambda\,\mathcal{D}\bar\lambda\,\mathcal{D}D\,\mathcal{D}P\,(\det g^2)\,(\det\partial_i^2)(\det\partial\!\!\!/)^2\,\delta(D)\delta(P)\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})\\ &\exp\left\{i\int d^3x\left[P^i\dot A_i+\dot D_1 P_1-\frac{g^2}{2}(P_i)^2-iC^{0i}P_i\bar\lambda\bar\lambda-\frac{1}{4g^2}(\partial_i A_j-\partial_j A_i)^2\right.\right.\\ &\left.\left.-\frac{i}{g^2}\lambda\partial\!\!\!/\bar\lambda+\frac{1}{2g^2}D^2-\frac{i}{2g^2}C^{ij}(\partial_i A_j-\partial_j A_i)\bar\lambda\bar\lambda\right]\right\}.\end{aligned} \tag{4.24}$$

In the exponent we recognize the first order lagrangian of the original theory (3.3) after integrations over Π_1^α, $\bar{\Pi}_{1\dot{\alpha}}$, Π_2^α and $\bar{\Pi}_{2\dot{\alpha}}$ are performed in its path integral.

Let us adopt the normalization to write partition function of non–anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory as

$$Z_{NA} = \int \mathcal{D}A_i\, \mathcal{D}P_i\, \mathcal{D}\lambda\, \mathcal{D}\bar{\lambda}\, \mathcal{D}D\, \mathcal{D}P\, \delta(D)\delta(P)\, \delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})$$
$$\exp\left\{\frac{i}{\hbar}\int d^3x\left[P^i\dot{A}_i + \dot{D}P - \frac{g^2}{2}(P_i)^2 - iC^{0i}P_i\bar{\lambda}\bar{\lambda}\right.\right.$$
$$\left.\left.-\frac{1}{4g^2}(\partial_iA_j - \partial_jA_i)^2 - \frac{i}{g^2}\lambda\partial\!\!\!/\bar{\lambda} + \frac{1}{2g^2}D^2 - \frac{i}{2g^2}C^{ij}(\partial_iA_j - \partial_jA_i)\bar{\lambda}\bar{\lambda}\right]\right\}. \quad (4.25)$$

Therefore, by the applying the transformation (2.9), partition function of its dual can be obtained as

$$Z_{NAD} = \int \mathcal{D}A_i\, \mathcal{D}P_i\, \mathcal{D}\lambda\, \mathcal{D}\bar{\lambda}\, \mathcal{D}D\, \mathcal{D}P\, \delta(D)\delta(P)\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})$$
$$\exp\left\{\frac{i}{\hbar}\int d^3x\left[P^i\dot{A}_i + \dot{D}P - \frac{1}{2g^2}(P_i)^2 - \frac{ig^2}{2}C_D^{0i}P_i\bar{\lambda}\bar{\lambda}\right.\right.$$
$$\left.\left.-\frac{4g^2}{4}(\partial_iA_j - \partial_jA_i)^2 - ig^2\lambda\partial\!\!\!/\bar{\lambda} + \frac{g^2}{2}D^2 - \frac{ig^4}{2}C_D^{ij}(\partial_iA_j - \partial_jA_i)\bar{\lambda}\bar{\lambda}\right]\right\}. \quad (4.26)$$

Here, we omitted the label "D" of the dual fields.

Comparing (4.18) and (4.24) which are obtained from the partition function for parent action (4.9), by integrating different set of fields, one concludes that the partition functions of non–anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory Z_{NA} and its dual Z_{NAD} are equivalent:

$$Z_{NA} = Z_{NAD}.$$

Therefore, under the strong-weak duality non–anticommutative $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory is invariant.

Loop corrections can be taken into account in terms of two different procedures. One of them is to calculate loop contributions to parent action and deduce the resulting theories from the loop corrected parent action. The other is to take into consideration loop corrections to $N = \frac{1}{2}$ supersymmetric $U(1)$ gauge theory [17] and then trying to formulate its dual. In the latter formulation it seems that our results survive with one loop corrected $C_{\mu\nu}$.

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